§6.7 Inner Product Spaces

<u>Goal</u>: Generalize the notion of the dot product in $\mathbb{R}^n$ to something similar in any vector space.

<u>Definition</u>: An <u>inner product</u> on vector space V is a function $\langle \, , \, \rangle$ such that for any two vectors u, v in V, $\langle u, v \rangle$ is a real number and satisfies the following:

For any vectors u, v, and w in V and scalar c

1) $\langle u, v \rangle = \langle v, u \rangle$

2) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

3) $\langle cu, v \rangle = c\langle u, v \rangle$

4) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$

A vector space with an inner product is called an <u>inner product space</u>.

Example

Consider the following inner product on $\mathbb{R}^2$:

If $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ define

$$\langle u, v \rangle = 2u_1 v_1 + 3u_2 v_2$$

a) show this an inner product

b) compute $\langle u, v \rangle$ if $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

solution:

a) check properties (1)-(4) hold. (exercise)

b) $\langle u, v \rangle = 2(1 \cdot 3) + 3(1 \cdot 5)$

$= 6 + 15$

$= 21$

Example

For distinct real numbers $t_0, \dots, t_n$ and p, q polynomials in $\mathbb{P}_n$ we can define the inner product

$$\langle p, q \rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n)$$

a) Verify this is an inner product (exercise)

Hint: For property (4) use that the only polynomial of degree that vanishes at $n+1$ points is the zero polynomial.

b) suppose $t_0 = -1$ $t_1 = 0$ $t_2 = 1$

Compute $\langle p, q \rangle$ where $p = 1 + t^2$ $q = 5 + t$

Solution:

$$\begin{aligned} \langle p, q \rangle &= p(-1)q(-1) + p(0)q(0) + p(1)q(1) \\ &= 2 \cdot 4 + 1 \cdot 5 + 2 \cdot 6 \\ &= 8 + 5 + 12 \\ &= 25 \end{aligned}$$

<u>Definition</u>: Let V be an inner product space with inner product $\langle \ , \ \rangle$.

1) For vector v in V, the <u>length</u> or <u>norm</u> of v is $\|v\| = \sqrt{\langle v, v \rangle}$

2) A unit vector is a vector u in V with length 1, i.e. $\|u\| = 1$

Just as before, unit vectors can be obtained by dividing a vector by its length

3) The <u>distance</u> between u and v is $\|u - v\|$

4) u and v are orthogonal if $\langle u, v \rangle = 0$.

<u>Warning!!!</u>

The choice of inner product matters. For instance if u, v are vectors in $\mathbb{R}^n$ and $u \cdot v = 0$, this doesn't mean $\langle u, v \rangle = 0$ for another inner product.

Orthogonal Projections and Best Approximations

If V is an inner product space with inner product $\langle \ , \ \rangle$ and W is a subspace with orthogonal basis $\{v_1, \dots, v_m\}$

The orthogonal project of vector y (in V) onto W is

$$\hat{y} = \text{proj}_W y = \frac{\langle y, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle y, v_2\rangle}{\langle v_2, v_2\rangle} v_2 + \cdots\cdots + \frac{\langle y, v_m\rangle}{\langle w_m, v_m\rangle} v_m$$

and $\hat{y}$ is the vector in W such that $\|y-\hat{y}\|$ is as small as possible.

- In other words, $\hat{y}$ is the "best approximation" of y by vectors in W.

- Just as before $y-\hat{y}$ is orthogonal to W.

Example

View $\mathbb{P}_1$ as a subspace of $\mathbb{P}_3$ and define inner product ~~as~~ by evaluation at $-2, -1, 1, 2$

(i.e. $\langle p, q \rangle = p(-2)q(-2) + \cdots + p(2)q(2)$)

a) verify $\{1, t\}$ is an orthogonal basis of $\mathbb{P}_1$

b) Find the best approximation of $t^3 + 2t^2$ by polynomials in $\mathbb{P}_1$

Solution:

a) $\langle 1, t \rangle = 1 \cdot (-2) + 1 \cdot (-1) + 1 \cdot 1 + 1 \cdot 2 = 0$

b) $$\operatorname{proj}_{\mathbb{P}_1}(t^3 + 2t^2) = \frac{\langle t^3 + 2t^2, 1 \rangle}{\langle 1, 1 \rangle} 1 + \frac{\langle t^3 + 2t^2, t \rangle}{\langle t, t \rangle} t$$

$$= \left(\frac{0 \cdot 1 + 1 \cdot 1 + 3 \cdot 1 + 16 \cdot 1}{1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 + 1 \cdot 1} \right) \cdot 1 + \left(\frac{0 \cdot -2 + 1 \cdot -1 + 3 \cdot 1 + 16 \cdot 2}{-2 \cdot -2 + -1 \cdot -1 + 1 \cdot 1 + 2 \cdot 2} \right) t$$

$$= \frac{20}{4} \cdot 1 + \frac{34}{10} t$$

$$= 5 + \frac{17}{5} t$$